Ph.D. COMPREHENSIVE EXAM – COMPLEX ANALYSIS – AUGUST 2005

Notation: $D=\{z\in\mathbb{C}: |z|<1\}$, $H(D)$ is the set of functions holomorphic in D.
Note: Five completed questions will ensure a passing grade.

1. Let $f\in H(D)$, with f not identically zero. Suppose that for every positive integer n there exists $g\in H(D)$ such that $f=g^n$. Show that f has no zero in D.

2. For $z\in\mathbb{C}\setminus[0,1]$, let

$$f(z)=\int_0^1 \frac{x}{x-z}\,dx.$$

a) Find the Laurent expansion of $f(z)$ valid in the annulus $\{z: |z|>1\}$, and determine its region of convergence.

b) Find an explicit form for $f(z)$.

3. Suppose that f is analytic in a simplyconnected domain G, which contains distinct points $z_1, z_2, \dots, z_n$. If

$$w(z)=\prod_{k=1}^{n}(z-z_k),$$

and C is a simple closed curve containing all the points $z_1,\dots,z_n$ inside, prove that

$$P(z):=\frac{1}{2\pi i}\int_C \frac{f(t)}{w(t)}\,\frac{w(t)-w(z)}{t-z}\,dt,$$

is a polynomial of degree $n-1$, interpolating $f(z)$ at $z_1,\dots,z_n$ (i.e. $P(z_k)=f(z_k)$, $k=1,\dots,n$).

4. Let $\mathfrak{F}$ be the family of all analytic maps of $\{z: \mathrm{Re}(z)>0\}$ into D. Put

$$\alpha=\sup_{f\in\mathfrak{F}}|f''(1)|.$$

a) Find a non-constant member of $\mathfrak{F}$.

b) Find an upperbound for α.

c) Prove that there exists $g\in\mathfrak{F}$ such that $g''(1)=\alpha$.

5. Suppose $\Omega\subseteq\mathbb{C}$ is open, $f:\Omega\to D$ is a conformal equivalence (i.e. f is holomorphic, 1-1 and onto) and $g:\Omega\to D$ is holomorphic. Suppose $\beta\in\Omega$ such that $f(\beta)=g(\beta)=0$. Show that

$$|f'(\beta)|\geqslant|g'(\beta)|$$

(Hint: Schwarz lemma).

6. Show that

$$\prod_{1}^{\infty}\left(\frac{n^3-z}{n^3+z}\right)$$

defines a <u>bounded</u> analytic function on $\{z: \mathrm{Re}(z)>0\}$, vanishing only on the set $\{n^3: n=1,2,\dots\}$.