PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—August 2004

Notation: $B = \{z \in \mathbf{C}: |z| < 1\}$, the open unit disc

1. (a) Suppose that f is analytic in B. Show that $f = F'$ for some function F analytic in B.

   (b) Let $B' = B \setminus \{0\} = \{z \in \mathbf{C}: 0 < |z| < 1\}$. Define $f(z) = \frac{1}{z}$. Show that there does *not* exist a function F analytic in B' with $F' = f$.

   (c) Explain the difference between B and B' here, by filling in the blank in the following theorem: Suppose G is a connected open subset of the plane. Every analytic function in G is the derivative of an analytic function if and only if G is ____________________. (No proof required here, just a correct statement.)

2. Suppose f is entire and $|f(z)| \leq |z|^2$ for all z. Show that $f(z) = cz^2$ for some constant c with $|c| \leq 1$.

3. Let f and g be analytic functions on B so that $f(z)g(z) = 0$ for all $z \in B$. Show that either $f(z) = 0$ for all $z \in B$ or $g(z) = 0$ for all $z \in B$.

4. Suppose that the nonconstant function f is analytic in a connected open set G which contains a simple closed curve on which $|f(z)| = c$ for some constant c. Prove that there is at least one zero of f inside this curve, assuming that the inside of the curve is a subset of G.

5. Determine the number of roots of the equation

$$z^6 + z^3 + 5z^2 - 2 = 0$$

   in the region $\{z: 1 < |z| < 2\}$.

6. (a) Let u be a continuous function on the closure of B which is harmonic on B. Express $u(0)$ in terms of an integral along the unit circle. No proof is required.

   (b) Let u be as in part (a). If $0 < r < 1$ and θ is real give a formula for $u(re^{i\theta})$ in terms of an integral along the unit circle. Again, no proof is required.

   (c) Let ϕ be the function defined on the unit circle by $\phi(e^{it}) = 1$ if $\alpha \leq t \leq \beta$ and $\phi = 0$ otherwise. Here α and β are constants with $0 \leq \alpha < \beta \leq 2\pi$. Using ϕ in place of u in the integral you found in part (b) defines a function on B. What is the value of that function at 0?