PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—January 2004

Notation: $B = \{z \in \mathbf{C}: |z| < 1\}$, the open unit disc

1. Let Ω be a connected open set in $\mathbf{C}$ and f a function analytic on Ω. (Note that Ω need not be simply connected.) Show that if $f = F'$ for some function F analytic on Ω then

$$\int_\gamma f(z)\,dz = 0$$

for every closed contour γ in Ω.

2. Suppose that f is a nonconstant entire function and that $|f(z)| \le 1 + |z|$ for all z. Show that f is one-to-one.

3. (a) Suppose that f is analytic on B and $\mathrm{Re}(f') > 0$ in B. Show that f is one-to-one in B.
Hint: One way is to show that $\mathrm{Re}\left(\dfrac{f(z) - f(w)}{z - w}\right) > 0$ for $z, w \in B$ with $z \neq w$.

   (b) Give an example of a function f analytic on B such that f' has no zero in B but f is *not* one-to-one in B.

4. Let γ be the simple closed curve defined by $\gamma(t) = e^{2\pi i t}$ for $0 \le t \le 1$, and let f be entire.

   (a) Suppose there exists a positive integer n such that

$$\int_\gamma \frac{f(z)}{(z - \alpha)^n}\,dz = 0$$

   for all $\alpha \in B$. Prove that f is a polynomial.

   (b) Suppose that for each $\alpha \in B$ there exists a positive integer $n(\alpha)$ such that

$$\int_\gamma \frac{f(z)}{(z - \alpha)^{n(\alpha)}}\,dz = 0.$$

   Prove that f is a polynomial.

5. Let Ω be a connected open set in $\mathbf{C}$, and fix a point $p \in \Omega$. Let $\mathcal{F}$ be the family of functions analytic on Ω so that

$$\sup\{|f(z)|: z \in \Omega\} \le 5$$

   for all $f \in \mathcal{F}$. Show that if M is defined by

$$M = \sup\{|f'(p)|: f \in \mathcal{F}\}$$

   then $M < \infty$, and that in fact the supremum is attained, that is, there exists $f \in \mathcal{F}$ so that

$$|f'(p)| = M.$$

6. (a) Give an example of a bounded harmonic function on B whose harmonic conjugate is unbounded.
Hint: It might be helpful to think in terms of conformal mappings.

   (b) Does there exist a nonconstant bounded harmonic function on $\mathbf{C}$?