PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—August 2003

Notation: $B = \{z \in \mathbf{C}\colon |z| < 1\}$, the open unit disc

1. Let f be an entire function with power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that a_n is real for all n if and only if $f(x)$ is real whenever x is real.

2. In this problem we define the path γ by $\gamma(t) = e^{it}$ on $[0, 2\pi]$. Suppose that f is analytic on an open set containing the closure of B, and let L denote the length of the image of γ under f. Show that

$$L \geq 2\pi |f'(0)|.$$

3. Suppose that f is analytic on $\{z\colon |z| > 1\}$.

   (a) Show that if $\lim_{z\to\infty} f(z) = 0$ then $\lim_{z\to\infty} f'(z) = 0$.

   (b) Show that if $\lim_{z\to\infty} f(z) = 42$ then $\lim_{z\to\infty} f'(z) = 0$.

4. Suppose that f is an entire function so that $f(0) = 0$, $f(\pi) = 0$, and for all $z \in \mathbf{C}$ we have $f(z+2\pi) = f(z)$ and $|f(z)| \leq \exp|\text{Im } z|$. Show that there exists a constant c so that $f(z) = c \sin z$ for all $z \in \mathbf{C}$.

5. Does there exist an analytic function f on $\mathbf{C} \setminus \{0\}$ so that

$$|f(z)| \geq |z|^{-1/2}$$

   whenever $z \in \mathbf{C} \setminus \{0\}$?

6. Let $\mathcal{F}$ denote the family of functions f for which there exists an open set containing the closure of B on which f is analytic and so that

$$\int_0^{2\pi} |f(e^{it})| dt \leq 1.$$

   Show that $\mathcal{F}$ is a normal family.