PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—January 2003

Notation: $B = \{z \in \mathbf{C} : |z| < 1\}$, the open unit disc

1. (a) Does there exist an analytic function f on B so that

$$f(1/n) = f(-1/n) = 1/n^2$$

whenever n is an integer satisfying $n \geq 2$?

(b) Does there exist an analytic function g on B so that

$$g(1/n) = g(-1/n) = 1/n^3$$

whenever n is an integer satisfying $n \geq 2$?

2. Let Ω be a connected open set in $\mathbf{C}$.

(a) For which functions f analytic on Ω is $|f|^2$ harmonic on Ω?

(b) Show that if f is analytic and nonzero on Ω then $\log|f|$ is harmonic on Ω.

3. Given that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$, compute $\int_0^\infty \sin(x^2)\,dx$.
Hint: Consider $\lim_{R\to\infty} \int_{C_R} e^{iz^2}\,dz$, where C_R describes the closed curve from 0 to R along the real axis, then from R to $Re^{i\pi/4}$ along the circle $|z| = R$ in the first quadrant, then from $Re^{i\pi/4}$ to 0 along a line segment.

4. Suppose that f is analytic on $D = \{z : \mathrm{Re}\, z > 0\}$ and satisfies $|f(z)| \leq 1$ for all $z \in D$. If $f(1) = 0$, how large can $f(2)$ be?

5. Suppose that f is analytic on B and that $|f(z)| < 1$ when $|z| < 1$. If f is not the identity function $f(z) \equiv z$, determine the largest number of fixed points f can have in B. That is, determine the largest number of solutions the equation $f(z) = z$ can have there.
Hint: First assume that 0 is a fixed point and apply the Schwarz lemma.

6. Let $\mathcal{F}$ denote the family of functions f analytic on B so that $f(0) = 0$ and

$$|f'(z)| \leq \frac{1}{1 - |z|}$$

for all $z \in B$. Show that $\mathcal{F}$ is a normal family.