PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—August 2002

Notation: $B = \{z \in \mathbf{C} : |z| < 1\}$, the open unit disc

1. Suppose that f is analytic on $\mathbf{C}$.

   (a) Assume that there exist positive constants a, b so that

   $$|f(z)| \leq a|z|^{1/2} + b$$

   for all z. Show that f is constant.

   (b) Assume now that there exist positive constants a, b so that

   $$|f(z)| \leq a|z|^{5/2} + b$$

   for all z. What can you conclude about f?

2. Suppose that f is a complex-valued function on B and that the functions $g = f^2$ and $h = f^3$ are both analytic on B. Show that f is analytic on B.

3. Determine the number of zeros of the polynomial

   $$z^7 - 4z^3 - 11$$

   in the region $\{z : 1 < |z| < 2\}$.

4. Determine whether

   $$\lim_{z \to 0} \left(\frac{1}{\tan^2 z} - \frac{1}{z^2} \right)$$

   exists as a finite number.

5. We say that two open sets in $\mathbf{C}$ are *conformally equivalent* if there is a one-to-one analytic function mapping one set onto the other. Suppose that V_1 and V_2 are nonempty connected and simply-connected open sets in $\mathbf{C}$, neither of which is conformally equivalent to B. Show that V_1 is conformally equivalent to V_2.

6. Let $f : B \to B$ be analytic, and assume that $f(0) = 0$. Show that, for $z \in B$,

   $$|f(z) + f(-z)| \leq 2|z|^2.$$

   Suggestion: Define

   $$g(z) = \frac{f(z) + f(-z)}{2z}$$

   if $z \in B^*$. (Here we write B^* for $B \setminus \{0\}$.) First use Schwarz's lemma to prove that $|g(z)| \leq 1$ for all $z \in B^*$, and then use Schwarz's lemma (along with a Taylor series for f) to show how the result follows.