

Notation: $B = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc

1. (a) Let Ω be an open set in \mathbb{C} and γ a smooth closed curve in Ω . Under what conditions on γ is it the case that $\int_{\gamma} f(z) dz = 0$ whenever f is a function analytic on Ω ?

(b) Give an example of an open set $\Omega \subset \mathbb{C}$, a function f analytic on Ω , and a smooth closed curve γ in Ω such that $\int_{\gamma} f(z) dz \neq 0$. Explain why this example does not contradict Cauchy's Theorem.

2. Suppose that f is a nonconstant analytic function on B and that a function g has an essential singularity at $f(0)$. Show that the composition $g \circ f$ is defined on $\{z : 0 < |z| < r\}$ for some $r > 0$ and has an essential singularity at 0.

3. Find all entire functions f satisfying the identity $f \circ f = f$.

4. Let $B^* = B \setminus \{0\}$, and let $f: B^* \rightarrow B^*$ be analytic. Show that f extends to an analytic map $g: B \rightarrow \mathbb{C}$ with $g(B) \subset B$.

5. Put $G = \{z : \text{Im } z > 0\}$, and let \mathcal{F} be the set of analytic functions f satisfying

$$f: G \rightarrow B, \quad f(i) = 0.$$

(a) Show that there exists $g \in \mathcal{F}$ satisfying

$$|g'(i)| = \sup \{|f'(i)| : f \in \mathcal{F}\}.$$

(b) Find all such $g \in \mathcal{F}$ explicitly.

6. Show that the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^3}\right)$$

converges uniformly on compact subsets of the complex plane.