PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—January 2001

Notation: $B = \{z \in \mathbf{C} : |z| < 1\}$, the open unit disc

1. (a) Let Ω be an open set in $\mathbf{C}$ and γ a smooth closed curve in Ω. Under what conditions on γ is it the case that $\int_\gamma f(z)\, dz = 0$ whenever f is a function analytic on Ω?

   (b) Give an example of an open set $\Omega \subset \mathbf{C}$, a function f analytic on Ω, and a smooth closed curve γ in Ω such that $\int_\gamma f(z)\, dz \neq 0$. Explain why this example does not contradict Cauchy's Theorem.

2. Suppose that f is a nonconstant analytic function on B and that a function g has an essential singularity at $f(0)$. Show that the composition $g \circ f$ is defined on $\{z : 0 < |z| < r\}$ for some $r > 0$ and has an essential singularity at 0.

3. Find all entire functions f satisfying the identity $f \circ f = f$.

4. Let $B^* = B \setminus \{0\}$, and let $f : B^* \to B^*$ be analytic. Show that f extends to an analytic map $g : B \to \mathbf{C}$ with $g(B) \subset B$.

5. Put $G = \{z : \operatorname{Im} z > 0\}$, and let $\mathcal{F}$ be the set of analytic functions f satisfying

$$f : G \to B, \quad f(i) = 0.$$

   (a) Show that there exists $g \in \mathcal{F}$ satisfying

$$|g'(i)| = \sup \{|f'(i)| : f \in \mathcal{F}\}.$$

   (b) Find all such $g \in \mathcal{F}$ explicitly.

6. Show that the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^3}\right)$$

   converges uniformly on compact subsets of the complex plane.