Algebra Comprehensive Exam

June 2018

Attempt all problems. Solving four problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used.

1. Let D_n be the dihedral group of order $2n$ and C_n be the cyclic group of order n. Recall that D_n may be presented as $\langle \sigma, \tau \mid \sigma^n = e,\ \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle$. Suppose that $n \geq 3$.

(a) Show that if $\psi \in \mathrm{Aut}(D_n)$ then $\psi(\sigma) = \sigma^k$ for some k. Use this to define a homomorphism $F : \mathrm{Aut}(D_n) \to \mathrm{Aut}(C_n)$.
(b) Show that F is onto and that the kernel of F is isomorphic to C_n.
(c) What is the order of $\mathrm{Aut}(D_n)$?

2. Explain what is meant by the conjugacy class of an element of a group. Describe all conjugacy classes in the symmetric group S_4 and in the alternating group A_4. You should explain the main facts that you are using to compile the lists, but you need not check every detail.

3. Let R be a non-zero commutative ring with 1. State Zorn's Lemma and then use it to prove that R has a maximal proper ideal.

4. State a theorem that classifies finitely-generated $\mathbb{Z}$-modules up to isomorphism. Explain how a finitely-generated $\mathbb{Z}/4\mathbb{Z}$-module gives rise to a finitely-generated $\mathbb{Z}$-module and use this to derive a theorem that classifies finitely-generated $\mathbb{Z}/4\mathbb{Z}$-modules up to isomorphism.

5. Let F be a field and H a finite subgroup of $F^\times$. Show that H is cyclic. (The symbol $F^\times$ denotes the group of non-zero elements of F with the operation of multiplication.)

6. Let $f(X) = X^6 + 3 \in \mathbb{Q}[X]$.

(a) Show that f is irreducible.
(b) Let θ be a root of f and $E = \mathbb{Q}(\theta)$. Show that E is a normal extension of $\mathbb{Q}$.
(c) Determine the Galois group of E over $\mathbb{Q}$. Be as explicit as possible.