Algebra Comprehensive Exam

January 2018

Attempt all problems. Solving four problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used.

1. Let $n \geq 2$ and $p \leq n$ be a prime number. Let P be a Sylow p-subgroup of the symmetric group S_n. Show that there is some $j \in \{1, \ldots, n\}$ such that $\sigma(j) = j$ for all $\sigma \in P$ if and only if p does not divide n.

2. Let p be a prime number and G a non-abelian group of order p^3. Show that the center of G has order p.

3. Let R be a commutative ring with 1, I and J ideals in R, and suppose that $I + J = R$. Show that the map $r + IJ \mapsto (r + I, r + J)$ is an isomorphism from R/IJ to $R/I \times R/J$.

4. Let R be a ring with 1, M a left R-module, and F a free left R-module. Suppose that $\varphi : M \to F$ is a surjective homomorphism. Show that there is a homomorphism $\psi : F \to M$ such that $\varphi \circ \psi = \mathrm{id}_F$.

5. State the Fundamental Theorem of Galois Theory. In addition to the theorem statement, you should also give the definitions of the major concepts that occur in the theorem.

6. Let ζ be a primitive seventh root of unity and $E = \mathbb{Q}(\zeta)$.

(a) Describe as explicitly as possible the Galois group of E over $\mathbb{Q}$.

(b) Show that E contains a unique quadratic field F and write F in the form $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Q}$.