Algebra Comprehensive Exam

January 2017

Attempt all problems. Solving five problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used. The hint is meant to guide you towards a possible solution to the problem, but your solution should make sense to someone who hasn't seen the hint.

1. Explain what it means for a group to be solvable. Show that the symmetric group S_4 is solvable.

2. Let p be a prime number and G be a group of order p^2. Show that G is abelian.

3. Let R be a commutative ring with 1. State Zorn's Lemma and then use it to prove that R has a maximal ideal.

4. Let R be a ring and

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} Q \longrightarrow 0$$

an exact sequence of left R-modules. Suppose that there is a homomorphism $h : Q \to M$ such that $g \circ h : Q \to Q$ is the identity map. Show that M is isomorphic to $K \oplus Q$.

5. Let p be a prime number. Show that the polynomial $x^p - x - 1 \in \mathbb{F}_p[x]$ is irreducible. Here $\mathbb{F}_p$ denotes the field with p elements. [Hint: It will be helpful to show that if θ is a root of the polynomial then $\theta + 1$ is also.]

6. Let E be a field, G a finite group of automorphisms of E, and

$$F = \{y \in E \mid \sigma(y) = y \text{ for all } \sigma \in G\}.$$

(a) Show that E is a normal extension of F.
(b) Show that E is a separable extension of F.