Algebra Comprehensive Exam

August 2016

Attempt all problems. Solving five problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used.

1. Show that the symmetric group S_5 has six Sylow 5-subgroups. Explain how to use this fact to define a transitive action of S_5 on the set $X = \{a, b, c, d, e, f\}$. Explicitly determine the isotropy subgroup (also known as the stabilizer) of some point $x \in X$. (You may choose which point to consider.)

2. Show that

$$\langle s, t \mid s^2,\ t^3,\ stst \rangle$$

is a presentation of the symmetric group S_3.

3. Let R be a commutative ring with 1. We say that R is Noetherian if it satisfies the ascending chain condition on ideals. State the ascending chain condition precisely and then show that R is Noetherian if and only if every ideal in R is finitely generated.

4. State a theorem that classifies finitely-generated $\mathbb{Z}$-modules up to isomorphism. Explain how a finitely-generated $\mathbb{Z}/6\mathbb{Z}$-module gives rise to a finitely-generated $\mathbb{Z}$-module and use this to derive a theorem that classifies finitely-generated $\mathbb{Z}/6\mathbb{Z}$-modules up to isomorphism.

5. Let F be a field and H a finite subgroup of $F^\times$. Show that H is cyclic. (The symbol $F^\times$ denotes the group of non-zero elements of F with the operation of multiplication.)

6. Let ζ be a primitive 12^{th} root of unity.

(a) Describe the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ explicitly. (There is no need to give a complete justification of your answer.)

(b) Organize the set $\{1, \zeta, \zeta^2, \zeta^3, \ldots, \zeta^{11}\}$ into orbits under the action of the Galois group.

(c) Explain how the answer to (b) can be used to determine the complete factorization of the polynomial $x^{12} - 1$ over $\mathbb{Q}$ and determine this factorization explicitly.