Algebra Comprehensive Exam

May/June 2016

Attempt all problems. Solving five problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used. The hints are meant to guide you towards a possible solution to the problems, but your solution should make sense to someone who hasn't seen the hint.

1. Explain what is meant by the conjugacy class of an element of a group. Describe all conjugacy classes in the symmetric group S_5 and in the alternating group A_5. You should explain the main facts that you are using to compile the lists, but you need not check every detail.

2. Show that, up to isomorphism, there is only one group of order 20 whose center is trivial. (Hint: Let G be such a group. Begin by showing that a Sylow 5-subgroup of G must be normal. Conclude that G must be a semidirect product and analyze the possibilities.)

3. State the Chinese Remainder Theorem. Illustrate the theorem by using it to express the ring

$$R = \frac{\mathbb{R}[T]}{(T^4 - T)}$$

explicitly as a product of fields.

4. Let R be a ring and

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} Q \longrightarrow 0$$

an exact sequence of left R-modules. Suppose that there is a homomorphism $h : Q \to M$ such that $g \circ h : Q \to Q$ is the identity map. Show that M is isomorphic to $K \oplus Q$.

5. State the Fundamental Theorem of Galois Theory. In addition to the theorem statement, you should also give the definitions of the major concepts that occur in the theorem.

6. Let ζ be a primitive 16^{th} root of unity.

(a) What is $[\mathbb{Q}(\zeta) : \mathbb{Q}]$? (It is fine just to quote an appropriate result.)

(b) Describe the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ explicitly.

(c) How many quadratic extensions of $\mathbb{Q}$ are contained in $\mathbb{Q}(\zeta)$? (A quadratic extension of $\mathbb{Q}$ is a field L such that $[L : \mathbb{Q}] = 2$.)