Algebra Comprehensive Exam

January 2016

Attempt all problems. Solving five problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used. The hints are meant to guide you towards a possible solution to the problems, but your solution should make sense to someone who hasn't seen the hint.

1. Let G be a finite group of even order greater than 2 whose Sylow 2-subgroups are cyclic. Show that G is not simple. (Hint: Consider the homomorphism $\varphi : G \to S_G$ defined by letting G act on itself by left multiplication, as in Cayley's Theorem. Here S_G is the symmetric group on the set G. Show that the image of φ is not contained in the alternating group A_G.)

2. Classify up to isomorphism all groups G of order 12 that contain an element of order 4. Give a presentation for a group in each isomorphism class.

3. Let R be a PID. Show that every irreducible element of R is prime. Give an example to show that the same does not necessarily hold if R is merely an integral domain.

4. Let R be a non-zero commutative ring with 1 and suppose that every R-module is free. Show that R is a field.

5. Let $p(x) = x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$. Show that p is irreducible, determine its Galois group over $\mathbb{Q}$, and explicitly determine all subfields of a splitting field for p.

6. Explain what it means for a polynomial to be solvable by radicals. Explain what it means for a group to be solvable. State a theorem that connects these two notions of solvability. (Be careful about the precise hypotheses.) Illustrate the theorem with an example of a polynomial that is solvable by radicals and an example of a polynomial that isn't.