Algebra Comprehensive Exam

August 2015

Attempt all problems. Solving five problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used. The hints are meant to guide you towards a possible solution to the problems, but your solution should make sense to someone who hasn't seen the hint.

1. Classify up to isomorphism all groups G of order 12 that contain an element of order 4. Give a presentation for a group in each isomorphism class.

2. Let G be a finite group, p a prime, and P a Sylow p-subgroup of G. Suppose that G has the property that if $H \neq G$ is a subgroup of G then $N_G(H)$ strictly contains H. Show that P is normal in G. (Here $N_G(H)$ denotes the normalizer of H in G. Hint: Show that $N_G(N_G(P)) = N_G(P)$.)

3. State the Chinese Remainder Theorem. Illustrate the theorem by using it to express the ring

$$R = \frac{\mathbb{Z}[x]}{(15, x^2+1)}$$

explicitly as a product of fields.

4. Identify each of the following tensor products of $\mathbb{Z}$-modules in simpler terms. It is not necessary to give all the details, but your solution should identify and address the essential steps in verifying that your response is correct.

(a) $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/5\mathbb{Z})$
(b) $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/5\mathbb{Z})$
(c) $(\mathbb{Z}/5\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/25\mathbb{Z})$

5. Let E be a normal extension of $\mathbb{Q}$ such that the Galois group of E over $\mathbb{Q}$ is isomorphic to Q_8, the quaternion group of order 8.

(a) Show that E contains a unique field of the form $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with $d_1, d_2 \in \mathbb{Q}^\times \setminus \left(\mathbb{Q}^\times\right)^2$ and $d_1 d_2 \notin \left(\mathbb{Q}^\times\right)^2$.

(b) Show that $d_1, d_2 > 0$. (Hint: Explain why complex conjugation gives rise to an element of the Galois group of E over $\mathbb{Q}$. Which element might it be?)

6. Let F be a field and $f \in F[x]$. Explain what is meant by the discriminant of f. Determine the discriminant of the polynomial $f(x) = x^4 + ax^2 + b$ in terms of a and b.