Algebra Comprehensive Exam

May/June 2015

Attempt all problems. Solving five problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used. The hints are meant to guide you towards a possible solution to the problems, but your solution should make sense to someone who hasn't seen the hint.

1. Let Q_8 be the quaternion group of order 8. Show that there is no group G such that $G/Z(G) \cong Q_8$. Here $Z(G)$ denotes the center of G. (Hint: It may help to recall the proof that there is no group G such that $G/Z(G)$ is a non-trivial cyclic group and to try a similar method.)

2. Show that

$$\langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$$

is a presentation of the symmetric group S_3.

3. Let $\mathbb{F}_3$ be the field with three elements and $V = \mathbb{F}_3^3$ be the 3-dimensional Euclidean space over $\mathbb{F}_3$ with standard basis e_1, e_2, and e_3. Let $T : V \to V$ be the linear map that satisfies $T(e_1) = e_2$, $T(e_2) = e_3$, and $T(e_3) = e_1$.

(a) Explain how T can be used to make V into an $\mathbb{F}_3[x]$-module.

(b) Explicitly determine all $\mathbb{F}_3[x]$-submodules of V.

4. Let $1 \leq m \leq n$ be integers, suppose that m divides n, and set $r = n/m$. Let $R = \mathbb{Z}/n\mathbb{Z}$ and $M = \mathbb{Z}/m\mathbb{Z}$.

(a) Briefly explain how M may be regarded as an R-module.

(b) Show that M is projective as an R-module if and only if the integers m and r are relatively prime.

(c) Under what condition is M a free R-module?

5. Explain what it means for a polynomial to be solvable by radicals. Explain what it means for a group to be solvable. State a theorem that connects these two notions of solvability. (Be careful about the precise hypotheses.) Illustrate the theorem with an example of a polynomial that is solvable by radicals and an example of a polynomial that isn't.

6. Let $F = \mathbb{F}_3(t)$, where $\mathbb{F}_3$ is the field with three elements and t is transcendental over $\mathbb{F}_3$. Let $f(x) = x^6 + x^4 + x^2 - t \in F[x]$. You may assume that f is irreducible in $F[x]$.

(a) Show that $f(-x) = f(x)$ and $f(x + 1) = f(x)$.

(b) Determine the Galois group of f over F. (Hint: Why is Part (a) there?)

(c) Let E be a splitting field of f over F. Describe all the fields that contain F and are contained in E.