Algebra Comprehensive Exam

January 2015

Attempt all problems. Solving five problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used. The hints are meant to guide you towards a possible solution to the problems, but your solution should make sense to someone who hasn't seen the hint.

1. Let Q_8 be the quaternion group of order 8.

(a) Recall that a Sylow 2-subgroup of S_4 is isomorphic to D_8, the dihedral group of order 8. Up to isomorphism, what is a Sylow 2-subgroup of S_6?

(b) What is the smallest value of n such that the symmetric group S_n has a subgroup isomorphic to Q_8? (Hint: Cayley's Theorem and Sylow's Theorems.)

2. A group G is said to be *supersolvable* if there is a chain

$$G = H_0 > H_1 > \cdots > H_{n-1} > H_n = \{e\}$$

such that each H_j is a normal subgroup of G and H_{j-1}/H_j is a cyclic group for $1 \leq j \leq n$. Let G be a finite supersolvable group and N a minimal non-trivial normal subgroup of G. Show that N is a cyclic group of prime order. (Hint: Begin by considering the largest j such that $H_{j-1} \cap N \neq \{e\}$.)

3. Let F be a field, E a finite extension of F, and L_1 and L_2 subfields of E containing F.

(a) Let

$$R = \left\{ \sum_{j=1}^{m} x_j y_j \,\middle|\, m \geq 1,\ x_j \in L_1,\ y_j \in L_2 \right\}.$$

Show that R is the smallest subfield of E that contains both L_1 and L_2.

(b) Let $S = L_1 \otimes_F L_2$ and make S into a ring with the usual addition operation and the multiplication operation that satisfies

$$(x \otimes y)(w \otimes z) = (xw) \otimes (yz)$$

for all $x, w \in L_1$ and $y, z \in L_2$. Show that S is a field if and only if $[R : F] = [L_1 : F]\,[L_2 : F]$. (Hint: Find a ring homomorphism from S onto R.)

4. Let R be an integral domain. If M is an R-module then we define the torsion submodule of M to be

$$T(M) = \{x \in M \mid rx = 0 \text{ for some non-zero } r \in R\}.$$

(a) Show that $T(M)$ is indeed a submodule of M.

(b) Show that if M is projective then $T(M) = \{0\}$. (Hint: First consider the case where M is free.)

5. Let F be a field. State a theorem that classifies finitely-generated modules over the polynomial ring $F[x]$ up to isomorphism.

6. Let $p(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 5. Let α be a root of $p(x)$ and suppose that $p(x)$ has the factorization $p(x) = (x - \alpha)q_1(x)q_2(x)$ in $\mathbb{Q}(\alpha)[x]$, where $q_1(x)$ and $q_2(x)$ are irreducible quadratic polynomials in $\mathbb{Q}(\alpha)[x]$. Show that the Galois group of $p(x)$ over $\mathbb{Q}$ is isomorphic to the dihedral group of order 10.