Algebra Comprehensive Exam

August 2014

Attempt all problems. Solving five problems completely correctly will guarantee a pass. The problems are intended to allow you to demonstrate your knowledge of algebra. You should give enough detail that the reader can be sure that you clearly understood the issue in each problem, had a thorough knowledge of the background material, and correctly applied any major theorems that you used. The hints are meant to guide you towards a possible solution to the problems, but your solution should make sense to someone who hasn't seen the hint.

1. What familiar group G is presented by

$$\langle x, y \mid yxy^{-1}x^{-3},\ xyx^{-1}y^{-3} \rangle?$$

(Hint: The order of G is less than 16.)

2. Let G be a group of order 24. Suppose that the center of G has order two and that $G/Z(G)$ is isomorphic to the alternating group A_4.

(a) Show that G has a unique Sylow 2-subgroup P. (Hint: What are the Sylow 2-subgroups of A_4?)

(b) Let H be a Sylow 3-subgroup of G. Show that $G = HP$.

(c) Show that the order of $\mathrm{Aut}(P)$, the automorphism group of P, is divisible by 3.

3. Let $R = \mathbb{Q}[x, y]/J$, where J is the ideal generated by $y^2 - x^3 - 4$.

(a) Show that R is an integral domain. (Hint: Recall that $\mathbb{Q}[x, y]$ is a UFD.)

(b) Show that the ideal in R generated by $y + J$ is maximal.

(c) Determine the radical of the ideal in R generated by $y - 2 + J$.

4. Let R be a Noetherian commutative ring with 1 and M an R-module. Notice that M is not assumed to be Noetherian. Show that M has submodules Q and N such that $M = Q \oplus N$, Q is injective, and N has no non-zero injective submodules. (Hint: Zorn's Lemma and Baer's Criterion.)

5. State the Fundamental Theorem of Galois Theory. In addition to the theorem statement, you should also give the definitions of the major concepts that occur in the theorem.

6. Determine the Galois group of the polynomial $f(x) = x^4 - 7$ over each of the following fields.

(a) $\mathbb{Q}(i)$ where $i^2 = -1$,

(b) $\mathbb{Q}(\sqrt{7})$.