PhD Exam Algebra June 2014

Provide complete proofs to all your assertions except where explicitly instructed otherwise. All notation and terminology should be clearly defined, and all proofs should be complete and stated in grammatically correct language. Any major theorems used in your proofs should be stated precisely.

Students should attempt all problems. Reasonably complete solutions of 5 problems will guarantee a passing grade. Solutions of 4 problems with substantial partial work on the remaining two may also achieve a passing grade.

1. From basic definitions and principles, classify all groups of order 8 up to isomorphism. Prove your list is complete.

2. Let G be a group of order 168 which has a subgroup H of index 6. Prove that G is not simple.

3. Let R be a commutative ring with 1.

   (a) Define precisely what it means for an R-module to be projective.

   (b) Define precisely what it means for an R-module to be injective.

   (c) Prove or disprove: If every R-module is projective, then every R-module is injective.

4. For any commutative ring R, an R-module M is said to be 'faithful' if $ax = 0$ for all $x \in M$ implies $a = 0$ in R.

   (a) Construct the field K of fractions of an integral domain R. You should define the set K, define addition and multiplication on K, define 0 and 1 in K, and prove that nonzero elements are invertible in K. You do not need to check the other ring axioms. Finally, prove that R is embedded in K as a subring.

   (b) For R and K as in part (a), let α be an element of K. Prove that α is a root of a monic polynomial in $R[x]$ if and only if there exists a faithful $R[\alpha]$-module which is finitely generated as an R-module.

   (c) Give an example of a ring R and an element $\alpha \in K \setminus R$ satisfying the conditions in part (b).

5. Let K/F be a field extension.

   (a) Define precisely what it means for K/F to be normal.

   (b) Define precisely what it means for K/F to be separable.

   (c) Give an example of an extension of fields which is normal and inseparable.

   (d) Give an example of an extension of fields which is non-normal and separable.

6. Let p be a prime and $\mathbb{Q}$ the field of rational numbers. Assume $f(x) = x^p - a \in \mathbb{Q}[x]$ is irreducible. Determine the splitting field E and Galois group G of $f(x)$ over $\mathbb{Q}$.