Provide complete proofs to all your assertions except where explicitly instructed otherwise. All notation and terminology should be clearly defined, and all proofs should be complete and stated in grammatically correct language. Any major theorems used in your proofs should be stated precisely.

Students should attempt all problems. Reasonably complete solutions of 5 problems will guarantee a passing grade. Solutions of 4 problems with substantial partial work on the remaining two may also achieve a passing grade.

1. Determine the order of the group with presentation
$$G = \langle a, b \mid aba^{-1}b^{-4},\ a^{-2}b^{-1}ab \rangle.$$

2. (a) Define what a Sylow subgroup of a finite group is, and state the main theorems about Sylow subgroups.
   (b) Using the Sylow theorems if desired, classify the groups of order 12 up to isomorphism. Prove your list is complete.

3. (a) Define precisely what is meant by the statement "Let $(R, \mathfrak{m})$ be a commutative local ring."
   (b) Let $(R, \mathfrak{m})$ be a commutative local ring. Prove that the following three conditions are equivalent:
      i. If M is a finitely generated R-module and $\mathfrak{m}M = M$, then $M = 0$.
      ii. If M is a finitely generated R-module, $N \subset M$ is a submodule, and $N + \mathfrak{m}M = M$, then $N = M$.
      iii. If M is a finitely generated R-module and $\{m_1, \cdots, m_n\} \subset M$ are elements of M whose images $\{\overline{m_1}, \cdots, \overline{m_n}\}$ form a generating set for the R-module $M/\mathfrak{m}M$, then $\{m_1, \cdots, m_n\}$ form a generating set for M.
   (c) Prove that the conditions in part (b) hold.

4. Let R be a commutative ring with identity.
   (a) Prove that the tensor product of two free R-modules is free.
   (b) Prove that the tensor product of two projective R-modules is projective.

5. Let F be a finite field and K/F be a finite extension. Prove that K/F is Galois, and that $\mathrm{Gal}(K/F)$ is cyclic.

6. Construct a Galois extension of fields with Galois group D_8 (i.e. the dihedral group of order 8). Prove your example is correct.