Provide complete proofs to all your assertions except where explicitly instructed otherwise. All notation and terminology should be clearly defined, and all proofs should be complete and stated in grammatically correct language. Any major theorems used in your proofs should be stated precisely.

Students should attempt all problems. Reasonably complete solutions of 5 problems will guarantee a passing grade. Solutions of 4 problems with substantial partial work on the remaining two may also achieve a passing grade.

1. (a) (70%) Let G be an abelian group and H a subgroup such that G/H is isomorphic to $\mathbb{Z}$. Prove that G is isomorphic to $H \times \mathbb{Z}$.

   (b) (30%) Prove or disprove the statement in part (a) with $\mathbb{Z}$ replaced everywhere by C_3, the cyclic group of order 3.

2. (a) (20%) Define precisely the *alternating group* A_n of degree n, and state its order.

   (b) (50%) Explicitly describe all the conjugacy classes of A_5 in the language and notation of permutations, and give the number of elements in each class.

   (c) (30%) Use the information in part (b) to prove that A_5 is a simple group.

3. (a) (20%) Define precisely the terms Unique Factorization Domain (or "factorial ring") and Principal Ideal Domain (or "principal ring").

   (b) (20%) Give an example of a Unique Factorization Domain which is not a Principal Ideal Domain.

   (c) (60%) Let R be a Principal Ideal Domain. Define what it means for an element $p \in R$ to be *irreducible*. Prove that if p is irreducible and p is a divisor of ab, for some elements $a, b \in R$, then p divides a or p divides b.

4. Let A be a commutative ring, let E be any A-module, and let D be any $\mathbb{Z}$-module. (Here $\mathbb{Z}$ denotes the ring of ordinary integers.)

   (a) (20%) Define the natural A-module structure of $\mathrm{Hom}(E, D)$. (In this use of Hom, E and D are considered as $\mathbb{Z}$-modules.)

   (b) (80%) If E is a projective A-module and D is an injective $\mathbb{Z}$-module, prove that $\mathrm{Hom}(E, D)$ is an injective A-module.

5. Suppose K is a field of prime characteristic $p > 0$. For any $a \in K$ such that a has no p-th root in K, show that $X^{p^n} - a$ is irreducible over K for all positive integers n.

6. Let ω be a primitive cube root of unity in an algebraic closure of $\mathbb{Q}$ and let $K = \mathbb{Q}(\omega)$. Suppose for a, $b \in K$ the polynomial $f(x) = x^6 + ax^3 + b$ is irreducible over K. Prove that the Galois group of $f(x)$ over K is either C_6, the cyclic group of order 6, the symmetric group S_3, or a nonabelian group of order 18. You do not have to show each of these possibilities occurs for some a and b, but you must show there are no other possibilities.