Provide complete proofs to all your assertions except where explicitly instructed otherwise. All notation and terminology should be clearly defined, and all proofs should be complete and stated in grammatically correct language. Any major theorems used in your proofs should be stated precisely.

Select two problems from each section of three problems and write complete solutions. Reasonably complete solutions will guarantee a pass. Solution of at least one problem from each section and significant partial progress on the others may possibly achieve a passing grade.

Section 1

1. (a) State the structure theorem for finite abelian groups.
   (b) Define the *dual* of an abelian group.
   (c) Assuming the structure theorem for finite abelian groups, prove that the dual of a finite abelian group G is isomorphic to G.

2. Suppose G is a finite group with the following property: For each subgroup H of G, there is a homomorphism $f : G \to H$ such that f is the identity map on elements of H. Prove that G is a product of groups of prime order. Hint: Induct on the order $|G|$.

3. (a) What does it mean for a group to be *simple*?
   (b) How many order 7 elements are there in a simple group of order 168? Prove your answer.

Section 2

4. Let R be a commutative ring with a 1. We say an ideal $\mathfrak{a}$ of R is **reducible** if it can be represented as an intersection $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ of two ideals $\mathfrak{b}$ and $\mathfrak{c}$ neither of which equals $\mathfrak{a}$. We say $\mathfrak{a}$ is **irreducible** if it is not reducible.
   Prove that if R is Noetherian, then every ideal is a finite intersection of irreducible ideals.

5. (a) Let R be a commutative ring with identity. What does it mean for an element of R to be prime? What does it mean for an element of R to be irreducible?
   (b) Let k be a field, and let R be the subring of $k[x]$ consisting of all polynomials whose coefficient of x is zero. (That is, there is no degree one term.) Prove that x^2 and x^3 are irreducible but not prime in R.
   (c) Let R be as in part (b). Prove that the ideal I of R consisting of all polynomials with zero **constant** term is not a principal ideal of R.

6. (a) Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

   (b) Let F be a free module over ring R, and let

$$M' \xrightarrow{\alpha} M \longrightarrow 0$$

   be an exact sequences of R-modules. Prove that for any module homomorphism $f : F \to M$ there is a module homomorphism $h : F \to M'$ such that $f = \alpha \circ h$.

Section 3

7. (a) Let $\mathbb{F}_2$ denote the finite field of 2 elements. State, with proof, explicit irreducible polynomials $p(x)$ over $\mathbb{F}_2$ for which a root generates the extensions of degree 2, 3, and 4 of $\mathbb{F}_2$.

   (b) Prove that any finite extension $K/\mathbb{F}_2$ is Galois, and describe the Galois group of K over $\mathbb{F}_2$ as explicitly as possible.

8. (a) Prove that $x^6 - 2x^3 + 2$ is irreducible over $\mathbb{Q}$.

   (b) Prove the splitting field L of $x^6 - 2x^3 + 2$ over $\mathbb{Q}$ is a $\mathbb{Q}(i, \omega, \sqrt[3]{2})$ where ω is a nontrivial cube root of 1, and then explicitly determine the Galois group of $L/\mathbb{Q}$.

   (c) Using the Fundamental Theorem of Galois Theory, describe as explicitly as possible all degree 6 extensions $K/\mathbb{Q}$ contained in L.

9. Let V and W be vector spaces over a field K, and let $V \otimes W$ be the tensor product. If $v_1, \ldots, v_r$ are linearly independent elements of V and $w_1, \ldots, w_r$ are any elements of W, prove that if $\sum_{i=1}^{r} v_i \otimes w_i = 0$ in $V \otimes W$, then $w_1 = w_2 = \cdots = w_r = 0$.