PhD Exam | Algebra | May 2012

Provide complete proofs to all your assertions except where explicitly instructed otherwise. All notation and terminology should be clearly defined, and all proofs should be complete and stated in grammatically correct language. Any major theorems used in your proofs should be stated precisely.

Students should attempt all problems. Reasonably complete solutions of 5 problems will guarantee a passing grade. Solutions of 4 problems with substantial partial work on the remaining two may also achieve a passing grade.

1. (a) Define precisely what it means for a group to be solvable.

   (b) If N is a normal subgroup of a group G, prove that G is solvable if and only if both N and G/N are solvable.

   (c) Let V be a vector space of dimension n over a field K, and suppose V_1 is a subspace of V of dimension 1. Prove that the set of invertible linear maps $T : V \to V$ satisfying $T(v) - v \in V_1$ for all $v \in V$ is solvable.

2. Let C be a cyclic group of order 4 and let $H = C \times C$. Let G be the group of automorphisms of H.

   (a) Prove there is a homomorphism of G onto the symmetric group S_3.

   (b) Prove that G has a normal subgroup N of order 16.

   (c) Determine the number of Sylow 2-subgroups of G.

3. Let R be a commutative ring with identity such that every submodule of every free module is free. Prove that R is a principal ideal domain.

4. Suppose R and S are commutative rings with identity, and $f : R \to S$ is a ring homomorphism satisfying $f(1_R) = 1_S$. Let $P \subset S$ be a prime ideal.

   (a) Show that $f^{-1}(P)$ is a prime ideal.

   (b) If P is a maximal ideal, does it follow that $f^{-1}(P)$ is also a maximal ideal? Prove your answer.

5. (a) State as completely as possible the Fundamental Theorem of Galois Theory for finite Galois extensions.

   (b) Let K be the splitting field of $x^7 - 1$ over $\mathbb{Q}$. Prove there is a unique subfield F of K satisfying $[F : \mathbb{Q}] = 3$.

   (c) Find an explicit irreducible polynomial $p(x) \in \mathbb{Q}[x]$ whose splitting field over $\mathbb{Q}$ is F.

6. (a) Define what it means for an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ to be solvable by radicals over $\mathbb{Q}$.

   (b) Prove that $x^5 - 80x + 2$ is irreducible in $\mathbb{Q}[x]$.

   (c) Prove that $x^5 - 80x + 2$ is not solvable by radicals over $\mathbb{Q}$.