Algebra Comprehensive Exam
Oklahoma State University
Department of Mathematics
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General Instructions: Define your terminology, and explain your notation. If you require a standard result, such as one of the Sylow theorems, then state it before you use it; otherwise, give clear and complete proofs of your claims. All problems are of equal value. Four problems solved completely and correctly will guarantee a pass. Partial solutions will be considered on their merits.

Conventions: All rings are commutative with 1, and all modules are unitary ($1 \cdot a = a$). The rational numbers are denoted by $\mathbb{Q}$, and the real numbers are denoted by $\mathbb{R}$.

1. Prove that a group of order $992 = 2^5 \cdot 31$ is solvable.

2. Let $F(X)$ be the free group on a set X. Let Y be a subset of X, and let N be the smallest normal subgroup of $F(X)$ containing Y. Show that $F(X)/N$ is a free group.

3. Let R be a commutative ring with 1. Given an ideal $I \subset R$, define
$$\sqrt{I} = \{f \in R : f^n \in I \text{ for some positive integer } n\}.$$
An ideal $Q \subsetneq R$ is called *primary* if whenever $ab \in Q$, either $a \in Q$ or $b \in \sqrt{Q}$.
   (a) Prove if Q is primary, then $\sqrt{Q}$ is prime.
   (b) Prove that if $\sqrt{Q}$ is maximal, then Q is primary.

4. Let R be a commutative ring with 1, and let P be an R-module.
   (a) Give two equivalent characterizations of an R-module P being projective. You need not prove the equivalence.
   (b) If P and Q are projective R-modules, prove that $P \otimes_R Q$ is a projective R-module. (Hint: One definition of projective refers to free modules.)

5. Give an example of a polynomial $p(x) \in \mathbb{Q}[x]$ whose Galois group over $\mathbb{Q}$ is the cyclic group of order 3. You must demonstrate the correctness of your example.

6. Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{R}$.
   (a) Determine the minimal polynomial f of α over $\mathbb{Q}$.
   (b) Show that f splits over $E = \mathbb{Q}(\alpha)$.
   (c) Show that $\text{Gal}(E/\mathbb{Q})$ contains an element of order 4.