ALGEBRA COMPREHENSIVE EXAM
AUGUST 2009

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims. All problems are of equal value. You have approximately two hours for this exam. Four problems solved completely and correctly will guarantee a pass. Partial solutions will be considered on their merits.

Notation: The field of rational numbers is denoted $\mathbb{Q}$.

Problem 1. Let G be a group and H a cyclic normal subgroup of G. Show that every subgroup of H is a normal subgroup of G.

Problem 2. Show that there is a unique (up to isomorphism) group of order 35, and that it is abelian.

Problem 3. Let A be an integral domain. The set of all prime ideals of A is denoted Spec(A). For any $\mathfrak{p} \in \mathrm{Spec}(A)$, let $A_\mathfrak{p}$ denote the localization of A at $\mathfrak{p}$, which is identified naturally as a subring of the quotient field of A. Show that

$$A = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} A_\mathfrak{p}.$$

Problem 4. Let R be a ring, and let M and N be (left) R-modules.

(1) Explain what it means to say that M is an injective R-module.
(2) Show that $M \oplus N$ is injective if and only if both M and N are injective.

Problem 5. Let F be a field, and G a finite subgroup of the multiplicative group F^* of nonzero elements of F. Show that G is cyclic.

Problem 6. Compute the Galois group of the polynomial $(x^3 - 2)(x^2 - 5)$ over $\mathbb{Q}$.