ALGEBRA COMPREHENSIVE EXAM
JANUARY 2009

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims. All problems are of equal value. You have approximately two hours for this exam. Four problems solved completely and correctly will guarantee a pass. Partial solutions will be considered on their merits.

Problem 1. Let H and K be subgroups of finite index of a group G.

(1) Show that $H \cap K$ has finite index in G.
(2) Show that there is a normal subgroup N of finite index in G, such that $N \subset H$.

Problem 2. Let G be a group of order 175. Show that the 7-Sylow subgroup is contained in the center of G.

Problem 3. Let F be a field, and let $F[[X]]$ denote the ring of power series in one variable X over F. Describe all the ideals of $F[[X]]$, and hence show that $F[[X]]$ is a local principal ideal domain.

Problem 4. Let R be a ring, and let M be a left R-module.

(1) Define what it means for M to be projective.
(2) If N is a left R-module such that $M \oplus N$ is projective then show that M is projective.
(3) Give an example, with a rigorous justification, of a module which is not projective.

Problem 5. Let G be a group, and let $\sigma_1, \dots, \sigma_n$ be distinct homomorphisms of G into the multiplicative group $F^\times$ of nonzero elements of a field F. Suppose there are constants $c_1, \dots, c_n \in F$ such that

$$c_1\sigma_1(g) + c_2\sigma_2(g) + \cdots + c_n\sigma_n(g) = 0$$

for all $g \in G$. Prove that $c_1 = c_2 = \cdots = c_n = 0$.

Problem 6. Let $\zeta = e^{2\pi i/17}$. Let $F = \mathbb{Q}(\zeta)$, and let $K = \mathbb{Q}(\zeta + \zeta^{-1})$.

(1) Show that $F/\mathbb{Q}$ and $K/\mathbb{Q}$ are Galois extensions, and determine their Galois groups.
(2) For any field homomorphism of F into the field of complex numbers, show that the image of K under this homomorphism is contained inside the field of real numbers.