Ph.D. Algebra Comprehensive Examination
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General Instructions: Define your terminology and explain your notation. If you require a standard result, such as one of the Sylow Theorems, then state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merits.

1. Let A be a finite abelian group. Show that there is an isomorphism of abelian groups (i. e., an isomorphism of $\mathbb{Z}$-modules)

$$A \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

(Hint: Show it first when A is a cyclic group.)

2. A subgroup H of a group G is called *special* if for each pair $x, y \in G$ with $x \notin H$ there exists a unique $u \in H$ such that $y^{-1}xy = u^{-1}xu$. Suppose H is a special subgroup of G.
(a) If $x \in G$, $x \notin H$, prove that $G = C(x)H$ and $C(x) \cap H = \{e\}$, where $C(x)$ is the centralizer of x in G.
(b) Prove that H is normal in G.

3. Let R be a commutative ring with 1. Two ideals I, J are called *relatively prime* if I, J are proper ideals and $I + J = R$.
(a) Let I, J be relatively prime ideals and let m, n be positive integers. Prove that I^m, J^n are relatively prime.
(b) Let I, J be relatively prime ideals and put $K = I \cap J$. Prove that there is an isomorphism of rings

$$R/K \cong R/I \times R/J.$$

4. Show that the element $3 \otimes 1$ equals zero in $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$ but does not equal zero in $3\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$.

5. (a) Suppose that the polynomial $f(x) = x^3 + a_2x^2 + a_1x + a_0 \in \mathbb{Q}[x]$ has one real root and two nonreal roots. Show that its discriminant is < 0.
(b) Let $f(x)$ be as in part (a). Show that its Galois group over $\mathbb{Q}$ is either S_2 or S_3.
(c) Find the discriminant of $f(x) = x^3 - x + \frac{1}{3}$.
(d) Let $f(x)$ be as in part (c). Find its Galois group over $\mathbb{Q}$.

6. Let L/K be an algebraic extension of fields and let $\phi : L \to L$ be a homomorphism such that $\phi(x) = x$ for all $x \in K$. Show that ϕ is an automorphism of L.