Comprehensive Examination in Algebra
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General Instructions: Define your terminology and explain your notation. If you require a standard result, such as one of the Sylow Theorems, then state it before you use it; otherwise, give clear and complete proofs of your claims. The problems are of equal value. Four perfect solutions will guarantee a pass. Partial solutions will be considered on their merits.

1. Let G be a group of order 36. Show that G either has a normal subgroup of order 3 or a normal subgroup of order 9.

2. Let G be a finite group of order $|G|$. For $g \in G$, let $o(g)$ denote the order of g.

   (a) For $g \in G$, define $\sigma_g : G \to G$ by $\sigma_g(h) = gh$. Show that $g \mapsto \sigma_g$ is a homomorphism from G to the symmetric group on G. [Recall that the symmetric group on a set is the collection of all bijections from the set to itself with composition as the operation.]

   (b) Find a formula for the sign of the permutation σ_g of G in terms of $|G|$ and $o(g)$. [Hint: Consider the cycle structure of σ_g.]

3. Let A be a commutative ring with 1 and suppose that A contains a field F (sharing the same 1) as a subring. Assume that A has finite dimension as an F-vector space. Show that A has a finite number of maximal ideals. [Hint: Suppose to the contrary and let $\{M_j\}_{j=1}^{\infty}$ be a sequence of distinct maximal ideals. Consider the chain of ideals $M_1 \supset M_1M_2 \supset M_1M_2M_3 \supset \cdots$.]

4. Let A be an abelian group, B a subgroup of A, and $f : B \to \mathbb{Q}$ a homomorphism. Show that there is a homomorphism $F : A \to \mathbb{Q}$ such that $F(b) = f(b)$ for all $b \in B$. [Here $\mathbb{Q}$ denotes the rational numbers under the operation of addition.]

5. Give an example of a polynomial $p(x) \in \mathbb{Q}[x]$ whose Galois group over $\mathbb{Q}$ is the cyclic group of order 3. You must demonstrate the correctness of your example.

6. Let K be a finite non-normal extension of $\mathbb{Q}$ and suppose that K has no subfields other than itself and $\mathbb{Q}$. Show that the only automorphism of K is the identity map.