Comprehensive Examination in Algebra
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General Instructions: Define your terminology and explain your notation. If you require a standard result, such as one of the Sylow Theorems, then state it before you use it; otherwise, give clear and complete proofs of your claims. The problems are of equal value. Four perfect solutions will guarantee a pass. Partial solutions will be considered on their merits.

1. Let G be a group of order 28.

(a) Show that G contains a unique subgroup of order 7.

(b) Show that the center of G is not trivial.

2. Let S_n denote the symmetric group of order $n!$. Suppose that G is a subgroup of S_7 and $|G| = 6!$. Show that G is isomorphic to S_6.

[**Hint**: Consider the action of G on the set $\{1, \ldots, 7\}$. Show that G fixes some element and explain why this is sufficient.]

3. Let R be a UFD. Suppose that $I \lhd R[x]$ is a prime ideal such that $I \cap R = \{0\}$. Show that I is a principal ideal.

[**Hint**: Begin by considering a polynomial of least degree in I.]

4. Let R be a ring and M a non-zero cyclic left R-module. Show that M has a quotient that is a simple module.

[**Definitions**: A module M is *cyclic* if there is some $x \in M$ such that $M = Rx$; a module is *simple* if it is non-zero and has no submodules but itself and $\{0\}$. Note that the ring R need not be commutative and need not have a 1. **Hint**: Apply Zorn's Lemma to an appropriately chosen set of submodules of M.]

5. Let $\alpha = \sqrt{2+\sqrt{3}}$ and $L = \mathbb{Q}(\alpha)$.

(a) Show that $L/\mathbb{Q}$ is a normal extension.

(b) Determine $\mathrm{Gal}(L/\mathbb{Q})$. Give a complete justification for your answer.

(c) Find all proper subfields of L. Express each in the form $\mathbb{Q}(\beta)$, where the minimal polynomial of β over $\mathbb{Q}$ is given.

6. Prove that the complex numbers are an algebraically closed field. The only facts from analysis to which your proof may appeal are the following.

(1) A polynomial of odd degree with real coefficients has a real root.

(2) Every complex number has a square-root in the complex numbers.

[**Hint**: What do these facts say about the possible degrees of algebraic extensions of $\mathbb{R}$ and $\mathbb{C}$?]