Comprehensive Examination in Algebra
Oklahoma State University
Department of Mathematics
August 2006

General Instructions: Define your terminology and explain your notation. If you require a standard result, such as one of the Sylow Theorems, then state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merits.

1. Let p be a prime number, G a finite p-group, and N a non-trivial normal subgroup of G. Show that the intersection of N with the center of G is non-trivial.

2. A group G is called *residually solvable* if for every $g \in G - \{e\}$ there is a normal subgroup N of G such that $g \notin N$ and G/N is solvable. Show that a finite residually solvable group is solvable.

3. Let R be a commutative ring with 1 and suppose that
$$r = a_0 + a_1x + \cdots + a_nx^n \in R[x]$$
is a nilpotent element of $R[x]$. Show that a_i is a nilpotent element of R for all i.

4. Let R be a ring and Z be its center. A *derivation* is a map $D : R \to R$ satisfying $D(a+b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in R$.
   (a) Let $r \in R$. Show that the map $A_r : R \to R$ given by $A_r(a) = ar - ra$ is a derivation of R.
   (b) Let D be a derivation. Prove that $D(Z) \subset Z$.
   (c) Let D be a derivation and $e \in Z$ an idempotent. Prove that $D(e) = 0$. (Recall that an idempotent is an element that satisfies $e^2 = e$.)

5. Let ζ_{16} be a primitive 16^{th} root of unity, $K = \mathbb{Q}(\zeta_{16}^4)$ and $L = \mathbb{Q}(\zeta_{16})$. Determine, with justification, $\mathrm{Gal}(L/K)$.

6. Prove that the complex numbers are an algebraically closed field. The only facts from analysis to which your proof may appeal are the following.
   (1) A polynomial of odd degree with real coefficients has a real root.
   (2) Every complex number has a square-root in the complex numbers.

   [Hint: What do these facts say about the possible degrees of algebraic extensions of $\mathbb{R}$ and $\mathbb{C}$?]