# Ph.D. Algebra Comprehensive Examination
# Oklahoma State University
# Department of Mathematics
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**General Instructions:** Define your terminology and explain your notation. If you require a standard result, such as one of the Sylow Theorems, then state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will be considered on their merits.

1. Let $G$ be a group of order $175 = 5^2 \cdot 7$. Show that $G$ is abelian.

2. Let $G$ be a finite solvable group.

   (a) Let $H$ be a subgroup of $G$. Show that $H$ is solvable.

   (b) Let $N$ be a minimal normal subgroup of $G$. (That is, $N$ is a non-trivial normal subgroup of $G$ such that if $L \subset N$ is a normal subgroup of $G$ then either $L = N$ or $L = \{e\}$.) Show that $N$ is abelian and that there is a prime number $p$ such that $x^p = e$ for all $x \in N$.

3. Let $R$ be a commutative ring with 1 and $r \in R$ an element that is not nilpotent. Show that there is a prime ideal $P \lhd R$ that does not contain $r$.

4. Let $R$ be a commutative ring with 1. If $M$ is an $R$-module then let $M^* = \mathrm{Hom}_R(M, R)$ be its dual module. Recall that the operations in $M^*$ are defined by $(\phi + \psi)(m) = \phi(m) + \psi(m)$ and $(r\phi)(m) = r\phi(m)$ for $\phi, \psi \in M^*$, $r \in R$ and $m \in M$. We may define a map $F_M : M \to (M^*)^*$ by $F_M(m)(\phi) = \phi(m)$ for $m \in M$ and $\phi \in M^*$. You may assume that $F_M$ is a homomorphism for every module $M$.

   (a) If $Y$ is a free $R$-module of finite rank then show that $F_Y$ is an isomorphism.

   (b) If $M$ and $N$ are $R$-modules such that $M \oplus N$ is free of finite rank then show that $F_M$ is an isomorphism.

5. Let $F$ be a field and $E$ a Galois extension of $F$ of finite degree. Let $h(x) \in F[x]$ be an irreducible polynomial. Show that all the irreducible factors of $h(x)$ in $E[x]$ have the same degree. Give an example to show that this need not be true if $E$ is merely a finite extension of $F$.

6. Let $F$ be a field of characteristic zero and suppose that $F$ contains a primitive cube root of unity. Let $E$ be a Galois extension of $F$ such that $\mathrm{Gal}(E/F)$ is isomorphic to the cyclic group of order 3. Show that there is some $\theta \in E$ such that $\theta^3 \in F$ and $E = F(\theta)$.