Ph.D. Algebra Comprehensive Examination
Oklahoma State University
Department of Mathematics
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General Instructions: Define your terminology and explain your notation. If you require a standard result, such as one of the Sylow Theorems, then state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merits.

1. Explicitly show that the symmetric group S_4 is a solvable group and list its composition factors. Is S_4 nilpotent? Give a reason for your answer.

2. Let G be a group of order 105, $P < G$ a Sylow 5-subgroup and $Q < G$ a Sylow 7-subgroup.

(a) Show that either P or Q is a normal subgroup of G.

(b) Show that G has a subgroup of order 35.

3. Let R be a ring and M a left R-module.

(a) Explain what it means to say that M is projective.

(b) If N is a left R-module and $M \oplus N$ is projective then show that M is projective.

(c) Show that a finitely-generated module over a PID is projective if and only if it is free.

4. Let R be a Noetherian commutative ring with 1. Show that if M and N are Noetherian R-modules then $M \otimes_R N$ is a Noetherian R-module.

5. Let F be a field of characteristic zero, $a \in F$ and $f(x) = x^5 - a \in F[x]$. Suppose that $f(x)$ is irreducible and let K be a splitting field for $f(x)$. Determine, with proof, the possible values of $[K : F]$. Show that if $\mathrm{Gal}(K/F)$ is abelian then $[K : F] = 5$.

6. Let E be a field of characteristic zero and $\alpha \in E$. Suppose that there is some subfield of E that does not contain α. Let

$$X = \{K \mid K \text{ is a subfield of } E \text{ and } \alpha \notin K\}$$

and partially order X by inclusion.

(a) Give a careful argument to show that X has a maximal element. Clearly state any result from set theory that your argument requires.

(b) Let L be a maximal element of X. Show that $L(\alpha)$ is a finite normal extension of L and that $\mathrm{Gal}(L(\alpha)/L)$ is a cyclic group of prime order.