Ph.D. Algebra Examination
Oklahoma State University
Department of Mathematics
January, 2003

Instructions: You must provide complete proofs of all your assertions except where explicitly instructed otherwise. Proofs should be written in clear and grammatical language. All notation and terminology should be clearly defined.

1. Let G be a group of order $539 = 7^2 \cdot 11$. Show that the 11-Sylow subgroup of G is contained in the center of G.

2. Let G be a group and N a normal subgroup of G. Suppose that the center of N is trivial and that every automorphism of N is inner. Let H be the centralizer of N in G. (Recall that $H = \{g \in G \mid gx = xg \text{ for all } x \in N\}$.)
(a) Show that for each $g \in G$ there exists $n \in N$ such that $gxg^{-1} = nxn^{-1}$ for all $x \in N$.
(b) For g, n as in part (a), prove that $n^{-1}g \in H$.
(c) Prove that $G \cong N \times H$.

3. Let R be a commutative Noetherian ring with identity. Prove that every nonunit of R can be factored as a product of irreducible elements of R.

4. Let R be a commutative ring with identity.
(a) Show that the nilpotent elements of R form an ideal I.
(b) Show that I equals the intersection of all the prime ideals of R.

5. Let K be a field and let G be a finite multiplicative subgroup of K. Prove that G is cyclic.

6. Suppose that L/K is a Galois extension of fields and that $f(X) \in K[X]$ is an irreducible polynomial. Show that all the irreducible factors of $f(X)$ in $L[X]$ have the same degree.