Ph.D. Algebra Examination
Oklahoma State University
Department of Mathematics
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Instructions: You must provide complete proofs of all your assertions except where explicitly instructed otherwise. Proofs should be written in clear and grammatical language. All notation and terminology should be clearly defined.

1. (a) Show that a group of order 30 has a normal subgroup of order 15. G
   (b) Determine (with proof) how many non-isomorphic groups of order 30 there are.

2. Let G be a finite group and let p be the smallest prime dividing the order of G. Prove that any subgroup of G of index p is normal. G

3. Let R be a commutative ring and let M

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Prove that M_2 is Noetherian if and only if M_1 and M_3 are Noetherian.

4. Let S be a commutative ring with 1 and let R be a subring of S containing 1. M
   (a) Let M be an R-module. Explain how $S \otimes_R M$ may be regarded as an S-module.
   (b) Let I be an ideal in R and let I' denote the ideal of S generated by I. Show that

$$S \otimes_R (R/I) \cong S/I' \qquad \text{(as } S\text{-modules).}$$

5. Let $p(X) = X^4 - 4X^2 + 1 \in \mathbb{Q}[X]$ and denote by F a splitting field of $p(X)$ over $\mathbb{Q}$. Let $\alpha \in F$ be one of the roots of $p(X)$. F
   (a) Show that the Galois group of F over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and find the action of each element of the Galois group on α. (Hint: Note that α^{-1} is also a root of $p(X)$.)
   (b) List all subfields of F (other than $\mathbb{Q}$ and F) by writing each subfield in the form $\mathbb{Q}(\beta)$ for some β and giving the minimal polynomial of β over $\mathbb{Q}$.

6. (a) State the Fundamental Theorem of Galois Theory for finite extensions E/K. F
   (b) Let E/K be a finite Galois extension with Galois group G and let G' be the commutator subgroup of G. Let L be the fixed field of G'. Show that L is the maximal abelian extension of K contained in E. (I. e., L/K is a Galois extension with abelian Galois group and L contains every intermediate field L', $K \subseteq L' \subseteq E$, for which L'/K is a Galois extension with abelian Galois group.)