PhD Exam Algebra Summer 2001

Provide complete proofs to all your assertions except where explicitly instructed otherwise. All notation and terminology should be clearly defined and all proofs should be complete and grammatically correct.

1. (a) If H is a cyclic normal subgroup of G, then prove that every subgroup of H is normal in G.
   (b) Give an example to show that the conclusion of the statement in in the first part would not be true if H were not cyclic.
2. (a) Let R be a ring and define injective R-module.
   (b) Define divisible Abelian group.
   (c) Prove that a $\mathbb{Z}$-module is injective if and only if it is divisible as an Abelian group.
3. (a) Determine all Abelian groups of order 360.
   (b) Are there any non-Abelian groups of order 360? If not, say why not and if so, give an example.
4. Let R be a ring with 1 and I and J be left ideals of R.
   (a) Prove: $(I : r) = \{x \in R | xr \in I\}$ is a left ideal of R for each $r \in R$.
   (b) Prove: The left R modules R/I and R/J are isomorphic iff $I = (J : s)$ for some $s \in R$ such that $s + J$ generates R/J.
5. Let K be the splitting field of $x^4 - 11$ over $\mathbb{Q}$.
   (a) Find the Galois group of K over $\mathbb{Q}$.
   (b) Find all fields F such that $\mathbb{Q} \subseteq F \subseteq K$ and F is normal over $\mathbb{Q}$.
6. Let $K \subseteq E \subseteq F$ be a tower of fields such that $[F : E] = 2$ and $[E : K] = 2$. Also, let $f \in K[x]$ be minimum polynomial of $\alpha \in F$.
   (a) Prove: The $\deg(f) = 1$, 2 or 4.
   (b) Prove: The splitting field of f has degree 1, 2, 4, or 8 over K.