Ph.D. Algebra Examination
Oklahoma State University
Department of Mathematics
Fall, 2000

Instructions: You must provide complete proofs of all your assertions except where explicitly instructed otherwise. Proofs should be written in clear and grammatical language. All notation and terminology should be clearly defined.

1. Show that there are exactly 2 isomorphism classes of groups of order 55: the cyclic group of order 55 and a nonabelian group generated by 2 elements x, y satisfying $x^{11} = y^5 = e$ and $yx = x^3y$.

2. Let G be a group.

   (a) Define the **center** Z of G.

   (b) Define the **centralizer** $C(x)$ of an element x of G. Prove that $C(x)$ is a subgroup of G.

   (c) Prove that, for any element x of G which does not belong to Z, then $C(x)$ is a proper subgroup of G which contains Z as a proper subgroup.

   (d) For this part, let G be a finite nonabelian group, and let p be the smallest prime divisor of the order of G. Prove that $(G : Z) \geq p^2$.

3. Let $\mathbb{R}$ be the field of real numbers and $A = \mathbb{R}[x, y]$ the ring of polynomials in two variables x, y with coefficients in $\mathbb{R}$. Let I be the ideal generated by $x^2 + y^2 - 1$.

   (a) Prove that the quotient ring A/I is an integral domain.

   (b) Prove or disprove that A/I is a unique factorization domain.

4. Let R be a ring.

   (a) Define what it means for a module M over R to be **projective**.

   (b) Prove that a direct sum $M_1 \oplus \cdots \oplus M_n$ of R-modules is projective if and only if each M_i is projective.

5. Let E/F be a finite extension of fields.

   (a) Define precisely what it means for E/F to be a Galois extension.

   (b) State the Fundamental Theorem of Galois Theory for finite Galois extensions E/F.

   (c) For an integer $n \geq 3$, let ω be a primitive n-th root of unity, and let $K = \mathbb{Q}(\omega)$ be the corresponding extension of the rationals $\mathbb{Q}$. Prove that $K/\mathbb{Q}$ is Galois, and describe its galois group.

6. Find the Galois group of $x^{12} - 3$ over $\mathbb{Q}$.